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On the solution stability of variational inequalities

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Abstract In the present paper, we will study the solution stability of parametric variational conditions

$$0 \in f(\mu, x) + N_{K(\lambda)}(x),$$

where M and Λ are topological spaces, $f: M \times R^n \to R^n$ is a function, $K: \Lambda \to 2^{R^n}$ is a multifunction and $N_{K(\lambda)}(x)$ is the value at x of the normal cone operator associated with the set $K(\lambda)$. By using the degree theory and the natural map we show that under certain conditions, the solution map of the problem is lower semicontinuous with respect to parameters (μ, λ). Our results are different versions of Robinson's results [15] and proved directly without the homeomorphic result between the solution sets.

Keywords Solution stability · Parametric variational conditions · Variational inequality · Degree theory · Lower semicontinuity.

1 Introduction

One of the most important areas of nonsmooth analysis is the study of stability to perturbation of the solutions to parameterized problems. Since many first-order conditions for optimality can be expressed in the form of variational conditions, there are much interest in solution stability to parametric variational conditions.

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Let us assume that M and Λ are topological spaces, $f: M \times R^n \to R^n$ be a function and $K: \Lambda \to 2^{R^n}$ be a multifunction. The variational conditions involving the set $K(\lambda)$ and the function $f(\mu, \cdot)$ is the problem of finding $x = x(\mu, \lambda)$ satisfying

$$0 \in f(\mu, x) + N_{K(\lambda)}(x), \tag{1}$$

where $N_{K(\lambda)}(x)$ is the value at x of the normal-cone operator associated with the set $K(\lambda)$ and $(\mu, \lambda) \in M \times \Lambda$ are parameters. When the multifunction K is convex valued, (1) is called a parametric variational inequality. We will denote by $S(\mu, \lambda)$ the solution set of the problem (1) corresponding to (μ, λ). Let $S(\mu_0, \lambda_0)$ be a solution set of (1) corresponding to $(\mu_o, \lambda_0) \in \Omega \times \Lambda$ that is, if $x_0 \in S(\mu_0, \lambda_0)$ then

$$0 \in f(\mu_0, x_0) + N_{K(\lambda_0)}(x_0). \tag{2}$$

Our main concern is to investigate the behavior of $S(\mu, \lambda)$ when (μ, λ) vary around (μ_0, λ_0). This problem interested many authors in the last two decades. For the relevant literature of the problem we will review briefly some papers that have a close connection with the present work.

Robinson [17] gave an excellent survey of work on the stability of generalized equation by proving the implicit-function theorems. These results can be interpreted in terms of the stability to perturbations, in μ, of the equation

$$0 \in f(\mu, x) + G(x).$$

In [17], Robinson showed that if f has a strong approximation then the solution map is Lipschitz and directional differentiable.

By using the metric projection method of Dafermos [2], Yen [19] obtained a theorem on Hölder continuity of the solution to a parametric variational inequalities in Hilbert spaces for strongly monotone operators. The result was extend by Demokos [4] to the case of variational inequalities in reflexive Banach spaces, where the solution set is a singleton.

Levy and Rockafellar [9], Levy and Mordukhovich [11] consider parametric variational inequalities with parameters in both f and K. However, their stability results concentrate on computing the proto-derivative and coderivative of the solution sets with respect to parameters, though which one can obtain such properties as Lipschitz continuity or the Aubin condition.

Recently, Robinson [15] (see also Ref. [16]) has introduced a localized version of the so-called normal maps to study solution existence and solution stability of variational inequalities. Based on the normal map and the degree-theoretic method, Robinson has established a result on the solution stability of the variational conditions in finite dimensional space. This is an important result on the solution stability of variational conditions for the cases of nonconvex sets.

Before restating Robinson's results in Ref. [15] we will recall some notions and events of the degree theory which used by Ref. [15] for the establishment of results on the solution stability of variational conditions. The notions and events of the degree theory can be found in Refs. [1,3,6,7,12,20].

Let Ω be an open bounded set in R^n. We denote by $\partial\Omega$ the boundary of Ω and $\overline{\Omega}$ the closure of Ω. Let $C^1(\overline{\Omega}) = C^1(\Omega) \cap C(\overline{\Omega})$, where $C^1(\Omega)$ is the set of all continuously differentiable functions $\phi: \Omega \to R^n$ and $C(\overline{\Omega})$ is the set of all continuous functions on $\overline{\Omega}$.

For each $\phi \in C(\overline{\Omega})$ we put $\|\phi\| = \max_{x \in \overline{\Omega}} \|\phi(x)\|$.

We will denote by $\text{dist}(x, A)$ the distance form a point $x \in R^n$ to a set $A \subset R^n$.

If $\phi \in C^1(\overline{\Omega})$, $J_\phi(x) = \det(\text{grad}\phi(x))$ and $Z_\phi = \{x \in \overline{\Omega} : J_\phi(x) = 0\}$ which is called the crease of ϕ.

It is well known that if $\phi \in C^1(\overline{\Omega})$ and $p \notin \phi(Z_\phi)$ then the set $\phi^{-1}(p)$ is finite (see, for instance [12, Theorem 1.1.2]).

Definition 1.1 (*a*) Let $\phi \in C^1(\overline{\Omega})$ and $p \notin \phi(Z_\phi) \cup \phi(\partial\Omega)$. The degree of ϕ at p with respect to Ω is defined by

$$\deg(\phi, \Omega, p) := \sum_{x \in \phi^{-1}(p)} \text{sgn}(J_\phi(x)). \tag{3}$$

(*b*) Let $\phi \in C^1(\overline{\Omega})$ and $p \notin \phi(\partial\Omega)$ such that $p \in \phi(Z_\phi)$. We define the degree of ϕ at p with respect to Ω is the number $\deg(\phi, \Omega, q)$ for any $q \notin \phi(Z_\phi) \cup \phi(\partial\Omega)$ such that $|p - q| < \text{dist}(p, \phi(\partial\Omega))$.
(*c*) Let $\phi \in C(\overline{\Omega})$ and $p \in R^n \backslash \phi(\partial\Omega)$. We define $\deg(\phi, \Omega, p)$, the degree of ϕ at p with respect to Ω, to be $\deg(\psi, \Omega, p)$ for any $\psi \in C^1(\overline{\Omega})$ such that $|\psi(x) - \phi(x)| < \text{dist}(p, \phi(\partial\Omega))$ for all $x \in \overline{\Omega}$.

The following list summarizes some properties most frequently used.

Theorem 1.1 *Suppose that $\phi \in C(\overline{\Omega})$ and $p \notin \phi(\partial\Omega)$. Then the following properties hold:*

(*a*) (*Normalization*) *If $p \in D$ then* $\deg(I, D, p) = 1$, *where I is the identity mapping.*
(*b*) (*Existence*) *If* $\deg(\phi, D, p) \neq 0$ *then there is $x \in D$ such that $\phi(x) = p$.*
(*c*) (*Additivity*) *Suppose that D_1 and D_2 are disjoint open sets of D. If $p \notin \phi(\overline{D} \backslash (D_1 \cup D_2)$ then*
$$\deg(D, f, p) = \deg(\phi, D_1, p) + \deg(\phi, D_2, p).$$
(*d*) (*Homotopy invariance*) *Suppose that $H : [0,1] \times D \to R^n$ is continuous. If $p \notin H(t, \partial D)$ for all $t \in [0,1]$ then* $\deg(H(t, .), D, p)$ *is independent of t.*
(*e*) (*Excision*) *If D_0 is a closed set of D and $p \notin \phi(D_0)$ then* $\deg(\phi, D, p) = \deg(\phi, D \backslash D_0, p)$.

Here are some particular notational conventions used in the sequel. If S is a multifunction from a space X to a space Y, then the set $\Gamma_S := \{(x, y) \in X \times Y : y \in S(x)\}$ is called the graph of S. Let $y_0 \in S(x_0)$ and W be a neighborhood of (x_0, y_0). The localization of S to W is the multifunction S_W whose graph is $\Gamma_S \cap W$. We recall that the multifunction S is called lower semicontinuous at $\bar{x} \in X$ if for any open set V of Y such that $S(\bar{x}) \cap V \neq \emptyset$, there exists a neighborhood U of $\bar{x}$ satisfying $S(x) \cap V \neq \emptyset$ for all $x \in U$.

The following theorem is a main result in Ref. [15].

Theorem 1.2 ([15], Theorem 3.2) *Let X be an open subset of R^n and U be a topological space. Suppose that f is a continuous function from $U \times X$ to R^n and K is a continuous multifunction from U to R^n.*
Let $X_0 \subset X$, $U_0 \subset U$ and $Z_0 \subset R^n$ be neighborhoods of x_0, u_0 and $z_0 := x_0 - f(u_0, x_0)$ respectively. Let $\xi : U_0 \times X_0 \to R^n$ be a function defined by $\xi(u, x) = x - f(u, x)$. Assume that:

(i) the localization to $(U_0 \times Z_0) \times X_0$ of the multifunction taking $(u, z) \in U_0 \times X_0$ to $(I + N_{K(u)})^{-1}(z) \in R^n$ is a single-valued, continuous function π;

(ii) there exists an open neighborhood $Z_1 \subset Z_0$ *of* $z_0 = x_0 - f(u_0, x_0)$ *such that* z_0 *is the unique solution of* $f_\pi(u_0, z) = 0$ *in* $\overline{Z}_1$ *and*

$$\deg(f_\pi(u_0, .), Z_1, 0) \neq 0,$$

where $f_\pi : U_0 \times Z_0 \to R^n$ *defined by* $f_\pi(u, z) = f(u, \pi(u, z)) + z - \pi(u, z)$.

Let $X_1 = \xi^{-1}(Z_1)$ *and define multifunctions* $\hat{Z} : U_0 \to Z_1$, $\hat{X} : U_0 \to X_1$ *by*

$$\hat{Z}(u) = \{z \in Z_0 \mid \pi(u, z) \in X_0, f_\pi(u, z) = 0\} \cap Z_1,$$

$$\hat{X}(u) = \{x \in X_0 \mid \xi(u, x) \in Z_0, 0 \in f(u, x) + N_{K(u)}(x)\} \cap X_1.$$

Then $\hat{Z}(u_0) = \{z_0\}$, $\hat{X}(u_0) = \{x_0\}$ *and the multifunctions* $\hat{Z}$ *and* $\hat{X}$ *are lower semicontinuous at* u_0.

As it was mentioned, Theorem 1.2 was proved by using the normal map and a result on the homeomorphism between the solution set of variational inequalities and the solution set of the normal map. These tools played a key role in arguments of Ref. [15]. From this and the degree-theoretic method, he drew fairly strong conclusion on the existence and continuity of solution of perturbed problems when the unperturbed problem satisfy certain regularity requirements. However, the hypothesis of Theorem 1.2 which posed on the normal map f_π is not easy to check and in principle the solution set $\hat{X}(u)$ could be smaller than the original solution set.

One may ask whether these conditions can be relaxed and the normal map method can be replaced?

In the present paper, we wish to give different versions of Theorem 1.2 by other approach, that is the natural map. Namely, we will show that under certain conditions which are posed on the so-called natural map and the metric projection, the solution map of parametric variational inequalities is lower semicontinuous. In order to obtain such a result we will have to use some facts of the degree theory and the method which was employed in Ref. [15] as well as the structure of the natural map.

It is noted that our results were proved directly without using the homeomorphic result between the solution set of a variational inequality and the solution set of the normal map. Besides, the conditions posed on the natural map, in some circumstances, are easy to verify and manipulate. For this we will give an illustrative example of our results at the end of the paper.

2 Main results

In this section, we will establish some result on the lower semicontiuity of the solution map of problem (1).

We now return to problem (1) to study properties of the perturbed variational conditions defined by the set $K(\lambda)$ and the map $f(\mu, \cdot)$, namely,

$$0 \in f(\mu, x) + N_{K(\lambda)}(x), \tag{4}$$

where $N_{K(\lambda)}(x)$ is the value at x of the normal-cone operator associated with the set $K(\lambda)$. We assume throughout the paper that the sets $K(\lambda)$ are Clarke regular at point x, at which we can compute the normal cone, so that we have

$$N_{K(\lambda)}(x) = \{x^* \in R^n \mid \langle x^*, y - x\rangle \leq o(\|y - x\|)\ \forall y \in K(\lambda)\}.$$

Let $S(\mu,\lambda)$ be the solution set of (4) corresponding to parameters (μ,λ) and x_0 be a solution of (4) corresponding to $(\mu_0,\lambda_0) \in M \times \Lambda$, that is $x_0 \in S(\mu_0,\lambda_0)$. Put

$$\pi(\lambda,x) = \left(I + N_{K(\lambda)}\right)^{-1}(x)$$

and

$$F_\rho(\mu,\lambda,x) = x - \pi(\lambda, x - \rho f(\mu,x)),$$

where $\rho > 0$. In case $\rho = 1$ we put

$$F(\mu,\lambda,x) = x - \pi(\lambda, x - f(\mu,x)).$$

It is easily seen that, for each fixed $(\mu,\lambda) \in M \times \Lambda$, x is a solution of (4) if and only if

$$0 \in F_\rho(\mu,\lambda,x). \tag{5}$$

Moreover, if $K(\lambda)$ has convex values then (4) becomes a parametric variational inequality and we have

$$F_\rho(\mu,\lambda,x) = x - \Pi_{K(\lambda)}(x - \rho f(\mu,x)),$$

$$F(\mu,\lambda,x) = x - \Pi_{K(\lambda)}(x - f(\mu,x)).$$

Here $\Pi_{K(\lambda)}(z)$ is the metric projection of z onto $K(\lambda)$. The later is called the natural map (see [5], p. 83).

The following theorem gives a sufficient condition for the lower semicontinuity of the solution map of (4).

Theorem 2.1 *Let X_0, M_0 and Λ_0 be neighborhoods of x_0, μ_0 and λ_0 respectively. Let f be a continuous function from $M_0 \times X_0$ to R^n and K be a multifunction from Λ_0 to R^n. Assume that:*

(i) *there exists a open neighborhood Z_0 of $z_0 = x_0 - f(\mu_0,x_0)$ such that the localization to $(\Lambda_0 \times Z_0) \times X_0$ of the multifunction taking $(\lambda,z) \in \Lambda_0 \times Z_0$ to $(I + N_{K(\lambda)})^{-1}(z)$ is a single-valued, continuous function π;*
(ii) *there exist an open bounded neighborhood $X_1 \subseteq X_0$ of x_0 such that x_0 is the unique solution of $F(\mu_0,\lambda_0,x) = 0$ in $\overline{X}_1$ and* $\deg(F(\mu_0,\lambda_0,.),X_1,0) \neq 0$.

Then there exist a neighborhood M_1 of μ_0, a neighborhood Λ_1 of λ_0 and an open bounded neighborhood Ω of x_0 such that the following assertions are fulfilled:

(a) *$\hat{S}(\mu,\lambda) := S(\mu,\lambda) \cap \Omega$ is nonempty for every $(\mu,\lambda) \in M_1 \times \Lambda_1$ and $\hat{S}(\mu_0,\lambda_0) = \{x_0\}$;*
(b) *$\hat{S}$ is lower semicontinuous at (μ_0,λ_0).*

Before proving the theorem we give some comparisons between Theorem 1.2 and Theorem 2.1. In our theorem, the continuity of multifunction K is not required while this hypothesis was necessary in the proof of Theorem 1.2. Condition (i) of Theorem 1.2 is the same as condition (i) of Theorem 2.1. They require the existence of neighborhoods X_0 and Z_0 of x_0 and $z_0 := x_0 - f(\mu_0,x_0)$ so that the localization of $[I + N_{K(\lambda)}]^{-1}(\cdot)$ to $X_0 \times Z_0$ is single and continuous. However, condition (ii) of Theorem 2.1 is somewhat different from condition (ii) of Theorem 1.2. In Theorem 2.1, this condition is posed on the natural map but in Theorem 1.2 it is posed on the normal map. In order to verify condition (ii) of Theorem 1.2 we first have to establish the formula of f_π then we compute $\deg(f_\pi, Z_1, 0)$. Meanwhile, condition (ii) of Theorem

2.1 is checked by computing the degree $d(F, X_1, 0)$. In some circumstances, this verification may be easier because the formula of F is often simpler than the formula of f_π.

Proof of Theorem 2.1 We will use some techniques from [15] and the structure of the natural map to give a direct proof.

Since $x_0 - f(\mu_0, x_0) \in Z_0$, by the continuity of f, there exist a neighborhood $X_2 \subset X_0$ of x_0, a neighborhood $M_0' \subset M_0$ of μ_0 such that $x - f(\mu, x) \in Z_0$ for all $(\mu, x) \in M_0' \times X_2$. By (i), F is single and continuous on $M_0' \times \Lambda_0 \times X_2$.

Choose an open bounded neighborhood Ω of x_0 such that $\Omega \subset X_1 \cap X_2$. By excising $\overline{X}_1 \backslash \Omega$, we have from (*e*) of Theorem 1.1 that

$$\deg(F(\mu_0, \lambda_0, .), \Omega, 0) = \deg(F(\mu_0, \lambda_0, .), X_1, 0) \neq 0. \tag{6}$$

Moreover, for any $\omega \in \partial\Omega$ we have $F(\mu_0, \lambda_0, \omega) \neq 0$. This implies that there exists a $\delta_\omega > 0$ such that $0 \notin B(F(\mu_0, \lambda_0, \omega), \delta_\omega) := B_\omega$. By the continuity of $F(\cdot, \cdot, \cdot)$ there exist a neighborhood $U_\omega \subset M_0'$ of μ_0, a neighborhood $\Lambda_\omega \subset \Lambda_0$ of λ_0 and a neighborhood Q_ω of ω such that $F(\mu, \lambda, z) \in B_\omega$ for all $(\mu, \lambda, z) \in U_\omega \times \Lambda_\omega \times Q_\omega$. Since $\partial\Omega$ is compact and $\partial\Omega \subset \cup Q_\omega$, there are some $\omega_1, \omega_2, \ldots, \omega_n$ such that $\partial\Omega \subset \cup_{i=1}^n Q_{\omega_i}$. Put $M_1 = \cap_{i=1}^n U_{\omega_i}$, $\Lambda_1 = \cap_{i=1}^n \Lambda_{\omega_i}$; we shall show that M_1, Λ_1 and Ω satisfy the conclusion of the theorem. In fact, we fix any $(\mu, \lambda) \in M_1 \times \Lambda_1$. For $z \in \overline{\Omega}$ and $t \in [0, 1]$ we consider a homotopy $p_t(z) = (1 - t)F(\mu_0, \lambda_0, z) + tF(\mu, \lambda, z)$. Choose any $\overline{z} \in \partial\Omega$; then $\overline{z} \in Q_{\omega_i}$ for some i and hence $(\mu, \lambda) \in U_{\omega_i} \times \Lambda_{\omega_i}$. By the convexity of B_{ω_i}, $p_t(\overline{z}) = (1 - t)F(\mu_0, \lambda_0, \overline{z}) + tF(\mu, \lambda, \overline{z}) \in B_{\omega_i}$. Hence $p_t(\overline{z}) \neq 0$. This means that $0 \notin p_t(\partial\Omega)$. By (*d*) of Theorem 1.1 we have

$$\deg F(\mu, \lambda, \cdot), \Omega, 0) = \deg(F(\mu_0, \lambda_0, \cdot), \Omega, 0) \neq 0.$$

By (*b*) of Theorem 1.1, there exists $x = x(\mu, \lambda) \in \Omega$ such that $F(\mu, \lambda, x(\mu, \lambda)) = 0$. Hence $S(\mu, \lambda) \cap \Omega \neq \emptyset$ for all $(\mu, \lambda) \in M_1 \times \Lambda_1$. Moreover, by (*ii*) we get $S(\mu_0, \lambda_0) \cap \Omega = \{x_0\}$.

It remains to prove (*b*). Suppose that V is an open such that $\hat{S}(\mu_0, \lambda_0) \cap V \neq \emptyset$. Sine $\hat{S}(\mu_0, \lambda_0) = \{x_0\}$, $x_0 \in V$. By the boundedness of Ω, the set $G := V \cap \Omega$ is bounded and open. By excising $\overline{\Omega} \backslash G$, we obtain from (*e*) of Theorem 1.1 that

$$\deg F(\mu_0, \lambda_0, \cdot), \Omega, 0) = \deg F(\mu_0, \lambda_0, \cdot), G, 0) \neq 0. \tag{7}$$

For any $\omega \in \partial G$ we have $F(\mu_0, \lambda_0, \omega) \neq 0$. Hence there exists a $\theta_\omega > 0$ such that $0 \notin B(F(\mu_0, \lambda_0, \omega), \theta_\omega) := B_\omega'$. By the continuity of $F(\cdot, \cdot, \cdot)$ there exist a neighborhood $U_\omega' \subset U_1$ of μ_0, a neighborhood $\Lambda_\omega' \subset \Lambda_1$ of λ_0 and a neighborhood Q_ω' of ω such that $F(\mu, \lambda, z) \in B_\omega'$ for all $(\mu, \lambda, z) \in U_\omega' \times \Lambda_\omega' \times Q_\omega'$. Note that ∂G is compact. Hence there are some $\omega_1, \omega_2, \ldots, \omega_n$ such that $\partial G \subset \cup_{i=1}^n Q_{\omega_i}'$. Put $U_2 = \cap_{i=1}^n U_{\omega_i}'$, $\Lambda_2 = \cap_{i=1}^n \Lambda_{\omega_i}'$. By the similar argument as the proof of part (*a*) and using (7) we can show that

$$\deg(F(\mu, \lambda, \cdot), G, 0) = \deg F(\mu_0, \lambda_0, \cdot), G, 0) \neq 0$$

for all $(\mu, \lambda) \in U_2 \times \Lambda_2$. According to (*b*) of Theorem 1.1, there exists $x = x(\mu, \lambda) \in G$ such that $F(\mu, \lambda, x) = 0$. This means that

$$S(\mu, \lambda) \cap G = \hat{S}(\mu, \lambda) \cap V \neq \emptyset$$

for all $(\mu, \lambda) \in U_2 \times \Lambda_2$. Hence $\hat{S}$ is lower semicontinuous at (μ_0, λ_0). The proof is complete. □

The following theorem is another version of Theorem 1.2 and Theorem 2.1 with modified conditions.

Theorem 2.2 *Let X_0, M_0 and Λ_0 be neighborhoods of x_0, μ_0 and λ_0 respectively. Let f be a continuous function from $M_0 \times X_0$ to R^n and $K\colon \Lambda_0 \to 2^{R^n}$ be a multifunction. Assume that:*

(*i*) *there exists a open neighborhood Z_0 of x_0 such that the localization to $(\Lambda_0 \times Z_0) \times X_0$ of the multifunction taking $(\lambda, z) \in \Lambda_0 \times Z_0$ to $(I + N_{K(\lambda)})^{-1}(z)$ is a single-valued, continuous function π;*

(*ii*) *there exist an open bounded neighborhood $X_1 \subseteq X_0$ of x_0 and $\rho_0 > 0$ such that x_0 is the unique solution of $F_\rho(\mu_0, \lambda_0, x) = 0$ in $\overline{X}_1$ and $\deg(F_\rho(\mu_0, \lambda_0, .), X_1, 0) \neq 0$ for all $\rho \in (0, \rho_0]$.*

Then there exist a neighborhood M_1 of μ_0, a neighborhood Λ_1 of λ_0 and an open bounded neighborhood Ω of x_0 such that the following assertions are fulfilled:

(*a*) *$\hat{S}(\mu, \lambda) := S(\mu, \lambda) \cap \Omega$ is nonempty for every $(\mu, \lambda) \in M_1 \times \Lambda_1$ and $\hat{S}(\mu_0, \lambda_0) = \{x_0\}$;*
(*b*) *$\hat{S}$ is lower semicontinuous at (μ_0, λ_0).*

Proof Choose $\overline{\rho} \in (0, \rho_0]$ such that $x_0 - \overline{\rho} f(\mu_0, x_0) \in Z_0$. By the continuity of f, there exist a neighborhood $X_2 \subset X_0$ of x_0, a neighborhood $M_0' \subset M_0$ of μ_0 such that $x - \overline{\rho} f(\mu, x) \in Z_0$ for all $(\mu, x) \in M_0' \times X_2$. Consider $F_{\overline{\rho}}(\mu, \lambda, x)$ with $(\mu, \lambda, x) \in M_0' \times \Lambda_0 \times X_2$. By (i), $F_{\overline{\rho}}(\cdot, \cdot, \cdot)$ is single and continuous on $M_0' \times \Lambda_0 \times X_2$. Choose an open bounded neighborhood Ω of x_0 such that $\Omega \subset X_1 \cap X_2$. By excising $\overline{X}_1 \backslash \Omega$, we have from (*e*) of Theorem 1.1 and (*ii*) that

$$\deg(F_{\overline{\rho}}(\mu_0, \lambda_0, .), \Omega, 0) = \deg(F_{\overline{\rho}}(\mu_0, \lambda_0, .), X_1, 0) \neq 0. \tag{8}$$

We now apply arguments in the proof of Theorem 2.1 again, with F replaced by $F_{\overline{\rho}}$, to obtain the desired conclusion. □

In the above theorem, although condition (*ii*) is heavier than the previous condition, condition (*i*) has been relaxed. In many cases, for examples, if multifunction K is either pseudo-Lipschitz or prox-regular with compatible parametrization (see definitions below) then condition (*i*) of Theorem 2.2 is satisfied while condition (*i*) of Theorem 1.2 and Theorem 2.1 are not fulfilled in general.

So far, we have provided general results about solution stability of perturbed variational conditions, but in order to apply them one needs to verify their hypotheses. One of them is the localization of $(I + N_{K(\lambda)})^{-1}$ is a single, continuous map. For this requirement, we will give a class of sets $K(\lambda)$ so that condition (*i*) of Theorem 2.2 is fulfilled. We cite below some notion and properties of sets which are prox-regular with compatible parametrization (see, for instance [14,15,18]).

Definition 2.1 Let $x_0 \in K(\lambda_0)$ and $v_0 \in N_{K(\lambda_0)}$. The sets $K(\lambda)$ is said to be prox-regular in x at x_0 for v_0 with compatible parametrization by λ at λ_0 if there exist neighborhoods U_0, V_0 and X_0 of λ_0, v_0 and x_0, respectively, with $\rho \geq 0$ such that

$$\langle v, x' - x \rangle - (\rho/2)\|x' - x\|^2 \leq 0$$

whenever $(x, \lambda, v) \in X_0 \times U_0 \times V_0$, $x \in K(\lambda)$, $x' \in X_0 \cap K(\lambda)$ and $v \in N_{K(\lambda)}(x)$.

Actually, the above definition requires that the indicator function $i_{K(\lambda)}$ of $K(\lambda)$, is prox-regular in x at x_0 for v_0 with compatible parametrization by λ at λ_0 in the term of Ref. [10].

The next theorem provides information on the properties of the projector on a prox-regular set that also satisfies condition (*i*) of Theorem 2.2.

Theorem 2.3 ([14], Corollary 4) *Let X and U be open set of R^n and R^m respectively, and let K be a multifuction from U to R^n that is continuous on U. Let x_0 be a point of X and λ_0 of U such that $x_0 \in K(\lambda_0)$. Let $v_0 = 0$ and suppose that $K(\cdot)$ is prox-regular in x at x_0 for 0 with compatible parametrization by λ at λ_0. Then for each real number $\beta > 1$ there exist an open neighborhood U_0 of λ_0, a closed neighborhood X_0 of x_0 and a neighborhood Z_0 of $z_0 := x_0$, such that the localization to $(U_0 \times Z_0) \times X_0$ of the multifunction taking $(\lambda, z) \in U_0 \times Z_0$ to $(I + N_{K(\lambda)})^{-1}(z)$ is a single, continuous function π that coincides with the localization to $(U_0 \times Z_0) \times X_0$ of the multifunction taking $(\lambda, z) \in U_0 \times Z_0$ to metric projection $\Pi_{K(\lambda)}(z)$.*

It is noted that, although Theorem 2.3 can be applied to Theorem 2.2 so that condition (*i*) is satisfied, Theorem 2.3 may be not applicable to condition (*i*) of Theorem 1.2. The reason is that, Z_0 in Theorem 2.3 is a neighborhood of x_0 but Theorem 1.2 requires that Z_0 is a neighborhood of $z_0 := x_0 - f(\mu_0, x_0)$. Moreover, if K is pseudo-Lipschitz then condition (*i*) of Theorem 2.2 is automatically fulfilled. This explains why in some circumstances the conditions of Theorem 2.2 are easy to verify.

Recall that the multifunction K is said to be pseudo-Lipschitz at $(\lambda_0, x_0) \in \text{Graph}K$ if there exist a neighborhood V of λ_0, a neighborhood W of x_0 and a constant $k > 0$ such that

$$K(\lambda) \cap W \subset K(\lambda') + k\|\lambda - \lambda'\| B(0, 1) \quad \forall \lambda, \lambda' \in \Lambda \cap V.$$

In this case we have the following result.

Theorem 2.4 *Let $M_0 \subset R^m$ and $\Lambda_0 \subset R^k$ be neighborhoods of μ_0 and λ_0 respectively. Let $X_0 \subset R^n$ be a closed convex neighborhood of x_0, $f: M_0 \times X_0 \to R^n$ be a continuous function and $K: \Lambda_0 \to 2^{R^n}$ be a multifunction with closed convex values. Assume that:*

- (*i*) *the multifunction K is pseudo-Lipschitz at (λ_0, x_0);*
- (*ii*) *there exists an open bounded neighborhood X_1 of x_0 and $\rho_0 > 0$ such that x_0 is the unique solution of $F_\rho(\mu_0, \lambda_0, x) = 0$ in $\overline{X}_1$ and $\deg(F_\rho(\mu_0, \lambda_0, .), X_1, 0) \neq 0$ for all $\rho \in (0, \rho_0]$, where F_ρ is defined by*

$$F_\rho(\mu, \lambda, x) = x - \Pi_{K(\lambda) \cap X_0}(x - \rho f(\mu, x))$$

for $(\mu, \lambda, x) \in M_0 \times \Lambda_0 \times X_0$.
Then there exist a neighborhood M_1 of μ_0, a neighborhood Λ_1 of λ_0 and an open bounded neighborhood Ω of x_0 such that the following assertions are fulfilled:

(*a*) *$\hat{S}(\mu, \lambda) := S(\mu, \lambda) \cap \Omega$ is nonempty for every $(\mu, \lambda) \in M_1 \times \Lambda_1$ and $\hat{S}(\mu_0, \lambda_0) = \{x_0\}$;*
(*b*) *$\hat{S}$ is lower semicontinuous at (μ_0, λ_0).*

Proof We first notice that, since K is convex valued, (4) becomes a parametric variational inequality.

According to Lemma 1.1 in Ref. [19] (see also Ref. [8]), it follows from (*i*) that there exist a neighborhood $\Lambda_0' \subset \Lambda_0$ of λ_0, a neighborhood $Z_0 \subset X_0$ of x_0 and a constant $k_0 > 0$ such that

$$\|\Pi_{K(\lambda) \cap X_0}(z) - \Pi_{K(\lambda') \cap X_0}(z)\| \le k_0 \|\lambda - \lambda'\|^{1/2}$$

for all $\lambda, \lambda' \in \Lambda_0'$ and $z \in Z_0$. Let $\pi(\lambda, z) = \Pi_{K(\lambda)\cap X_0}(z)$. For any $z, z' \in Z_0$ and $\lambda, \lambda' \in \Lambda_0'$ we have

$$\begin{aligned}\|\pi(\lambda, z) - \pi(\lambda', z')\| &= \|\Pi_{K(\lambda)\cap X_0}(z) - \Pi_{K(\lambda')\cap X_0}(z')\| \\ &\leq \|\Pi_{K(\lambda)\cap X_0}(z) - \Pi_{K(\lambda)\cap X_0}(z')\| + \|\Pi_{K(\lambda)\cap X_0}(z') - \Pi_{K(\lambda')\cap X_0}(z')\| \\ &\leq \|z - z'\| + k_0\|\lambda - \lambda\|^{\frac{1}{2}}.\end{aligned}$$

Consequently, $\pi : \Lambda_0' \times Z_0 \to X_0$ is uniformly continuous on $\Lambda_0' \times Z_0$. Chose $\overline{\rho} \in (0, \rho_0]$ such that $x_0 - \overline{\rho} f(\mu_0, x_0) \in Z_0$. By the continuity of f, there exists a neighborhood M_0' of μ_0, a neighborhood X_2 of x_0 such that $x - \overline{\rho} f(\mu, x) \in Z_0$ for all $(\mu, x) \in M_0' \times X_2$. We now consider $F_{\overline{\rho}}$ which is defined by

$$F_{\overline{\rho}}(\mu, \lambda, x) = x - \Pi_{K(\lambda)\cap X_0}(x - \overline{\rho} f(\mu, x_0))$$

for $(\mu, \lambda, x) \in M_0' \times \Lambda_0' \times X_2$. By the above, $F_{\overline{\rho}}$ is continuous on $M_0' \times \Lambda_0' \times X_2$. Choose a bounded open neighborhood Ω of x_0 such that $\Omega \subset X_1 \cap X_2$. Then Ω is contained in the interior of X_0. By using the arguments as in the proof of Theorem 2.1 for $F_{\overline{\rho}}$ and Ω, we show that there exist a neighborhood M_1 of μ_0, a neighborhood Λ_1 of λ_0 so that for each $(\mu, \lambda) \in M_1 \times \Lambda_1$, the equation $F_{\overline{\rho}}(\mu, \lambda, x) = 0$ has a solution $x(\mu, \lambda)$ in Ω. As $x(\mu, \lambda)$ belongs to the interior of X_0, $x(\mu, \lambda)$ is a solution of (4). We now apply arguments in the proof of Theorem 2.1 again to obtain the desired conclusion. □

To end we give an illustrative example below.

Example 2.1 Let $M_0 = [-2, 2] \subset R$, $\Lambda_0 = [-1, 2] \subset R$ and $X_0 = R^2$. Let f: $M_0 \times R^2 \to R^2$ defined by

$$f(\mu, x) = (x_1^2 + \mu x_2, x_1), \quad x = (x_1, x_2)$$

and $K \colon \Lambda_0 \to 2^{R^2}$ defined by

$$K(\lambda) = \{(x_1, x_2) : 2x_1 - x_2 \leq 4, x_1 + x_2 = 2\lambda\}. \tag{9}$$

Put $(\mu_0, \lambda_0) = (0, 1)$, $x_0 = (1, 1)$ and $X_1 = B(x_0, 1)$. Then we have the following assertion:

(*a*) $x_0 = (1, 1)$ and $x_0' = (0, 2)$ are solutions of (4) at (μ_0, λ_0);
(*b*) conditions (*i*) and (*ii*) of Theorem 2.2 are satisfied;
(*c*) there exists an open bounded neighborhood Ω of x_0 such that the solution map $\hat{S}(\cdot) = S(\cdot) \cap \Omega$ is lower semicontinuous at (μ_0, λ_0).

In fact, we have $f(\mu_0, x) = (x_1^2, x_1)$ and

$$K(\lambda_0) = \{(x_1, x_2) : 2x_1 - x_2 \leq 4; x_1 + x_2 = 2\}.$$

Hence for every $x = (x_1, x_2) \in K(\lambda_0)$ we get

$$\langle f(\mu_0, x_0), x - x_0\rangle = \langle (1, 1), (x_1 - 1, x_2 - 1)\rangle = x_1 + x_2 - 2 = 0.$$

Consequently, x_0 is a solution of (4) at (μ_0, λ_0). Similarly, we also have $x_0' = (0, 2)$ is a solution of (4) at (μ_0, λ_0).

Since $K(\cdot)$ is convex valued and is Lipschitz continuous (see, for instance [13]), condition (*i*) of Theorem 2.2 and condition (*i*) of Theorem 2.4 are fulfilled. Here we can choose $Z_0 = B(x_0, \sqrt{2})$. It remains to show that x_0 is the unique solution of $F_\rho(\mu_0, \lambda_0, x) = 0$ in $\overline{X}_1$ and $\deg(F_\rho(\mu_0, \lambda_0, .), X_1, 0) \neq 0$ for all $\rho > 0$. We have

$$\begin{aligned} F_\rho(\mu_0, \lambda_0, x) &= (x_1, x_2) - \Pi_{K(\lambda_0)\cap X_0}[(x_1, x_2) - \rho f(\mu_0, (x_1, x_2))] \\ &= (x_1, x_2) - \Pi_{K(\lambda_0)\cap X_0}(x_1 - \rho x_1^2, x_2 - \rho x_1) \\ &= (x_1, x_2) - (1 + \frac{1}{2}(x_1 - x_2 - \rho x_1^2 + \rho x_1), 1 - \frac{1}{2}(x_1 - x_2 - \rho x_1^2 + \rho x_1) \\ &= \left(\frac{1}{2}(x_1 + \rho x_1^2 - \rho x_1 + x_2) - 1, \frac{1}{2}(x_1 - \rho x_1^2 + \rho x_1 + x_2) - 1\right). \end{aligned}$$

Hence $F_\rho(\mu_0, \lambda_0, x) = 0$ if and only if

$$\begin{cases} x_1 + \rho x_1^2 - \rho x_1 + x_2 = 2 \\ x_1 - \rho x_1^2 + \rho x_1 + x_2 = 2. \end{cases}$$

The above system gives two solutions $(1, 1)$ and $(0, 2)$. It is obvious that $x_0 = (1, 1)$ is the unique solution of the equation $F_\rho(\mu_0, \lambda_0, x) = 0$ in $\overline{X}_1$.

We now compute the degree $\deg(F_\rho(\mu_0, \lambda_0, .), X_1, 0)$. Since

$$J_{F_\rho} = \begin{vmatrix} \frac{1}{2}(1 - \rho + 2\rho x_1) & \frac{1}{2} \\ \frac{1}{2}(1 + \rho - 2\rho x_1) & \frac{1}{2} \end{vmatrix},$$

it yields $J_{F_\rho}(x_0) = \frac{\rho}{2} > 0$. Hence $\deg(F_\rho(\mu_0, \lambda_0, \cdot), X_1, 0) = 1$.

Let $M_1 = (-\frac{1}{28}, \frac{1}{28})$ and $\Lambda_1 = (\frac{3}{4}, \frac{5}{4})$. Then for each $(\mu, \lambda) \in M_1 \times \Lambda_1$ we have

$$F_\rho(\mu, \lambda, x) = \frac{1}{2}((1 - \rho)x_1 + \rho x_1^2 + (1 + \mu\rho)x_2 - 2\lambda, (1 + \rho)x_1 - \rho x_1^2 + (1 - \mu\rho)x_2 - 2\lambda).$$

Hence $F_\rho(\mu, \lambda, x) = 0$ if and only if

$$\begin{cases} (1 - \rho)x_1 + \rho x_1^2 + (1 + \mu\rho)x_2 = 2\lambda \\ (1 + \rho)x_1 - \rho x_1^2 + (1 - \mu\rho)x_2 = 2\lambda. \end{cases}$$

This system gives two solutions

$$\left(x_1^0, x_2^0\right) = \frac{1}{2}\left(1 + \mu + \sqrt{(1 + \mu)^2 - 8\mu\lambda}, 4\lambda - 1 - \mu - \sqrt{(1 + \mu)^2 - 8\mu\lambda}\right)$$

and

$$\left(x_1', x_2'\right) = \frac{1}{2}\left(1 + \mu - \sqrt{(1 + \mu)^2 - 8\mu\lambda}, 4\lambda - 1 - \mu + \sqrt{(1 + \mu)^2 - 8\mu\lambda}\right).$$

Note that $(1 + \mu)^2 - 8\mu\lambda > 0$ for all $(\mu, \lambda) \in M_1 \times \Lambda_1$. Thus we obtain $S(\mu, \lambda) = \{(x_1^0, x_2^0), (x_1', x_1')\}$. Putting $x(\mu, \lambda) = (x_1^0, x_2^0)$ we get

$$\lim_{(\mu,\lambda)\to(0,1)} x(\mu, \lambda) = (1, 1) = x_0.$$

Choosing $\Omega = X_1$ we have $\hat{S}(\mu_0, \lambda_0) = S(\mu_0, \lambda_0) \cap \Omega = \{x_0\}$. Moreover, $\hat{S}(\mu, \lambda)$ is lower semicontinuous at (μ_0, λ_0).

We notice that Theorem 1.2 cannot apply to our example. In fact, we have $z_0 = x_0 - f(\mu_0, x_0) = (0, 0)$. Hence for any $\epsilon > 0$, $B(z_0, \epsilon) \nsubseteq B(x_0, \sqrt{2})$. This implies that the set Z_1 in Theorem 1.2, is not contained in Z_0. Consequently, conditions (*i*) and (*ii*) of Theorem 1.2 are invalid. □

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